ISSN 1064-2307, Journal of Computer and Systems Sciences International, 2011, Vol. 50, No. 3, pp. 491–498. © Pleiades Publishing, Ltd., 2011.
Original Russian Text © A.Yu. Perevaryukha, 2011, published in Izvestiya Akademii Nauk. Teoriya i Sistemy Upravleniya, 2011, No. 3, pp. 140–148.

SIMULATION

Uncertainty of Asymptotic Dynamics in Bioresource Management Simulation

A. Yu. Perevaryukha

St. Petersburg Institute of Information Science and Automation, Russian Academy of Sciences, Chetyrnadtsataya Liniya, 39, St. Petersburg, 199178 Russia

Received April 27, 2010

Abstract—Specific features of simulating a system that experiences structural changes in the course of its development are studied by the example of a controlled biological population. The problem of whether nonattracting chaotic sets can arise in dynamic systems with more than one attractor is considered in this context. The formalism of hybrid automata as applied to simulation problems of biological processes is described. Specific features of the phase portrait of the developed dynamic model are characterized by locally disconnected boundaries of basins of attraction of two attractors. The conclusion on limited predictability of the dynamics of some controlled natural systems is made, which is a consequence of uncertainty with respect to the motion of the system towards one of possible stable states.

DOI: 10.1134/S1064230711010151

INTRODUCTION

Management of renewable biological resources involves challenges related to prediction, optimization, and appropriate evaluation of their up-to-date state and reproduction ability. Despite a great number of works on optimal choice and control in ecological problems, striving for optimality entails significant risks in practice. In this work, we consider nonlinear phenomena associated with specific features of reproduction of biological populations and leading to negative consequences when their use is optimized to get maximum profit. In some cases, principles of optimal control theory used to study application models contradict other theoretical outcomes resulted from studying dynamic chaos. Once the controlled system reaches certain points in the parameter space, its dynamics can undergo a rather wide range of qualitative changes.

We describe one of these critical, previously overlooked changes associated with boundaries of basins of attraction. Analysis of the reasons of degradation of Caspian Sea sturgeon populations demonstrates the risk that arises once a complex ecological system is controlled under insufficient knowledge on the plant system properties and nonlinear nature of the processes.

1. THE PROBLEM OF ECOLOGICAL PROCESS SIMULATION

At the first stage of its development, the new scientific discipline borrowed methods of mathematical ecology from classical mechanics. Vito Volterra tried to apply the mathematical apparatus of integro-differential equations he worked out within the elastic theory to interaction of abstract competing populations [1]. However, his theory could not analyze any true statistical observation data since they were not available at that time. Its further development was reduced to modifying the terms in the right-hand sides of systems of ordinary differential equations. A number of significant factors were left aside the evolving methodology. A great number of complex systems and processes experience large-scale changes in the course of their development, which makes it difficult to describe their dynamics using ordinary differential equations with smooth right-hand sides.

Discrete dynamic systems are becoming the main mathematical technique applied in computer simulation of dynamics of real-world population processes. In many cases, it seems reasonable to represent natural processes by algorithmic models based on difference equations and matrices due to peculiarities of ecosystem life-cycle. Simulating algorithms used in such works include functional dependences and conditional operators of a programming language that impose restrictions and specify conditions for particular functional dependencies to be chosen. G. Paulic [2] who used FORTRAN and V.V. Menshutkin who implemented the simulating algorithms on ALGOL-60 were the first to use computers in ecological research.

Among other objectives, simulation is to provide the basis for prediction. We consider phenomena that result in theoretical impossibility of long-term prediction and accompany the research of nonlinear models of how biological processes develop. Simulating algorithms frequently use functional dependencies (actually they act as an evolution operator of the dynamic system) that can lead to topologically non-equivalent phase portraits to formalize critical cause–effect relations. However, there are also negative consequences caused by a number of nonlinear effects that have nothing to do with bifurcations of attractors associated with changes of values of model parameters.

2. DYNAMIC CHAOS IN MODELS OF DYNAMICS OF BIOLOGICAL PROCESSES

Involved in simulation in different fields, discrete dynamic systems are especially widely yet frequently unsuccessfully applied in mathematical description of biological processes.

W. Ricker [3] developed a theory that became widely known and is generally referred to as stock-recruitment in publications. The theory explains the complex nature of the chain of relations accompanying the entire process of fish population reproduction. Mortality in early ontogenesis of fish is high and controlled by natural factors as well as factors that depend on the initial spawning stock density. It can be clear cannibalism, predators reacting to large amounts of food, or mortality caused by starvation when food is limited. We carried out a number of laboratory experiments that demonstrated deceleration of growth of fish population given the high density.

Ricker [3] was first to propose mathematical formalization of the nonmonotonic dependence of recruitment on stock $R = f(S)$. However, along with his numerous followers, he did not consider this model in terms of theory of dynamic systems. A discrete dynamic system that describes population dynamics as functional iterations using the Ricker model $R_{j+1} = aR_j\exp(-bR_j)$ may yield topologically non-equivalent phase portraits. For the control parameter a successively increased to reach the bifurcation value, the dynamic system represented as the semigroup of iterations $\{\psi^{(j)}\}_{j \geq 0}$, where $R_0, R_1, R_2, \ldots$ is a series of points that describe evolution of the system and are given by the condition $R_{j+1} = \psi(R_j)$ for all $j \geq 0$, has the global attractor. When the transit mode ends, the system moves into the stable equilibrium state with the stationary point R^*, with the whole phase space being the basin of attraction Ω for R^* (true for attractors for any $a > 0$ yet biologically significant only for $a > 1$). J. Milnor [4] considered different interpretations of the concept of attractor and analyzed examples to come up with a generalized definition. In what follows, we use the Milnor's definition to study effects associated with changes of attractors.

When the derivative at the fixed point no longer meets the stability criterion, the system undergoes metamorphosis. For dynamic systems of the type involved, it happens when the condition $|\psi'(R^*)| < 1$ is not met for the first derivative, which follows from the Grobman-Hartman theorem [5]. The mapping acquires two new cyclic points $\psi^n(R^*) = \psi^{n+2}(R^*)$ that are the fixed points for the second iteration $\psi^2(R)$. When the control parameter changes within the range $a > \mathbf{e}^2$, we move to chaos through the infinite cascade of period-doubling bifurcations (each time for $\psi^{2n}{}'(R^*) = -1$). The interval of the value of the control parameter between two sequential bifurcations reduces rapidly as the cycle period grows. Once it passes through the cascade of doublings, the trajectory is attracted to the attractor that differs from the finite union of smooth submanifolds of the phase space and is called a "strange attractor". A chaotic mode emerges (Fig. 1); dynamics of changes looks stochastic. The main but not the only property of chaos is its sensitive dependence on initial conditions—there is an exponential recession of close trajectories. For tangent bifurcations, the chaotic range of the values a is broken by windows of periodicity with stable cycles of different periods. It includes the cycle of period 3 that emerges exactly according to the A.N. Sharkovskii theorem. The theorem proved before the concept of strange chaotic attractor was proposed and the very term "chaos" was introduced defines co-existence of cycles of different periods in the mapping. Three acts as the final number in the special Sharkovskii order [6]; if there is a cycle of period 3 in the dynamic system, there are also cycles of all other possible periods.

The order of changes in the behavior of the system trajectory represented as the doubling bifurcation cascade for successively increased parameter is known as the M. Feigenbaum scenario. The scenario is realized in nonlinear unimodal maps with the extremum close to quadratic $x_{n+1} = \lambda\sin(\pi x_n)$, $x_{n+1} = 4\lambda x_n(1 - x_n)$. Two universal constants—the Feigenbaum numbers [7] – characterize the rate of moving to chaos when the cycle period becomes infinite. The value of Schwartz derivative

$$S_\psi = \frac{\psi```(R)}{\psi`(R)} - \frac{3}{2}\left(\frac{\psi``(R)}{\psi`(R)}\right)^2 \qquad (2.1)$$

acts as the criterion of moving to chaos through the bifurcation cascade in the mappings of the type involved.

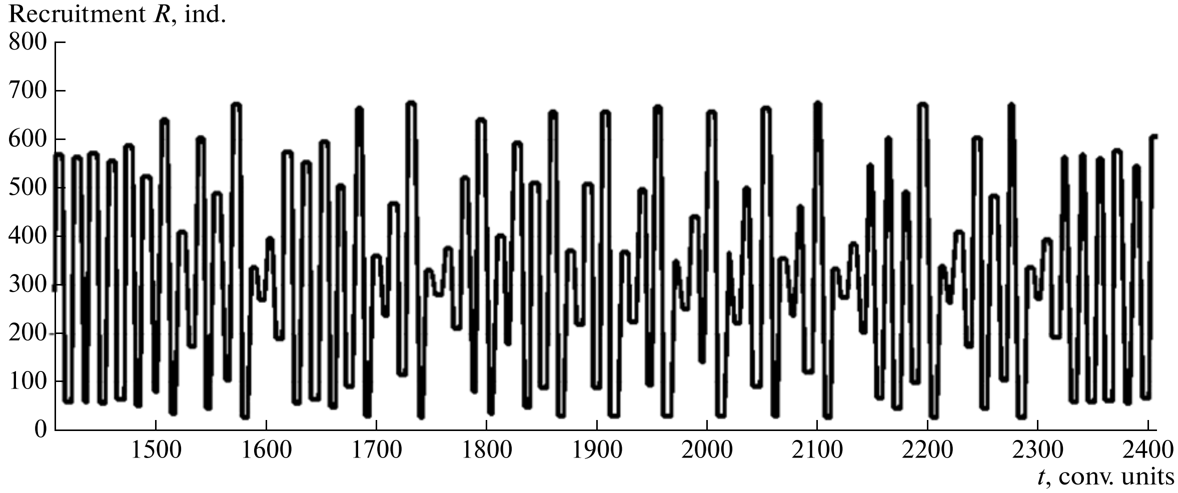

Fig. 1. Chaos in the model based on the Ricker formula.

This value (generally called Schwartzian) possesses an important property, viz. it preserves the sign as the function $\psi(\psi(...\psi(R)...)) \equiv \psi^n(R)$ is calculated during n iterations. The strictly negative value of the differential Schwartz invariant $S_\psi < 0$ is the condition for the Feigenbaum scenario to be realized in the unimodal mapping.

It is interesting that a similar scenario of moving to chaos is established for another model within the stock-recruitment theory proposed by J. Sheperd. While in the Ricker model this happens when the parameter a that characterizes the system self-recovery rate is increased, the Sheperd model had the bifurcation parameter introduced to take into account the influence of environmental resistance factors that makes recovery slow down. The two models proposed in the same subject field turn out to have opposite behavior interpretations. Application of nonlinear dynamics methods has shown that the models similar in the form of dependence are mutually contradictory.

3. STRUCTURAL CHANGES OF STRANGE ATTRACTORS OF DISCRETE MAPPINGS

Aperiodic motion of the trajectory points as the property of some one-dimensional unimodal maps was first mentioned in the course of analyzing the results of R. May's research in computer simulation of biological problems. Further studies have shown that application of this mathematical apparatus is connected with a whole number of other nonlinear phenomena. Only part of them have been fully studied up to now, nonlinear dynamics that employs computational methods being one of the most rapidly developing discipline of modern mathematics.

Several critical effects used to evaluate adequacy of the models proposed in different disciplines are connected with possible windows of periodicity that emerge after tangent bifurcations. For such bifurcation, the derivative of the n-th iteration at new fixed points is $\psi^{n\prime}(R^*) = 1$. The cycle of period 3 is formed when the third iteration $\psi^3(R)$ of the Ricker function for $a = 22.54$ gets six new fixed points, which are intersections with the bisectrix $R_n = R_{n+1}$, with three of them forming a stable cycle and the other three, unstable (Fig. 2).

Unlike the doubling bifurcation that changes only the attractor, the tangent bifurcation results in lost stability since it changes more than just a type of the attractor of the dynamic system. It is topology of basins of attraction that changes as well, which specifies the way the trajectory behaves in the window of periodicity, viz. within a range of values of the control parameters $\Delta a = a_c - a_t$. Contrary to the widespread simplified knowledge on dynamics of the models involved, the behavior of the trajectory for $a \in [a_c, a_t]$ is not bounded by stable cyclic modes.

At the instant of moving to the window of periodicity, basins of attraction of the strange and regular attractors cross and the trajectory of the dynamic system, through an interval of chaotic motion, tends to the subset of the phase space consisting of cyclic points. We need to consider the mapping $R_{n+1} = \psi^3(R_n)$ as an independent dynamic system. The mapping has three attractors within the range of values Δa of existence of the window of periodicity, with the boundaries of their basins of attraction becoming fractal by the definition given in [8]. Such structure of the boundaries results in the fact that the subset of the

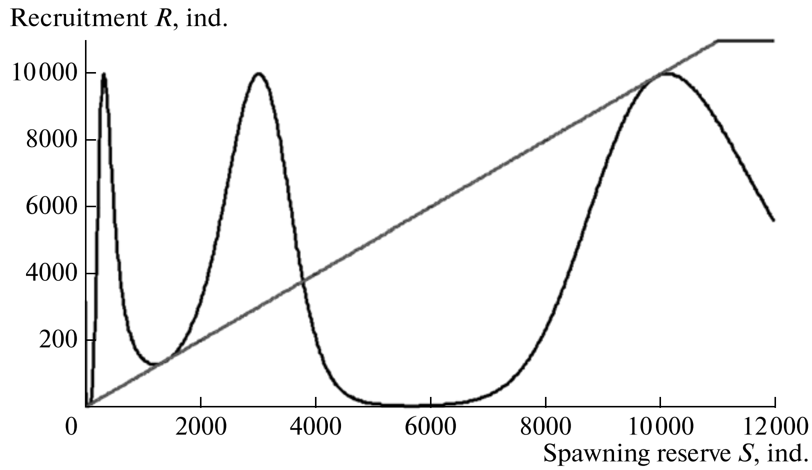

Fig. 2. Instant of the tangent bifurcation.

phase space, along which the trajectory moves, is characterized by the fractional topological dimension. The so-called box-counting dimension

$$d = \lim_{\delta \to 0} \frac{\ln Z(\delta)}{\ln(1/\delta)} < 1, \quad (3.1)$$

where $Z(\delta)$ is the minimum number of cells of the size δ necessary to cover the boundary of the basin of attraction of the attractor [9], is frequently used to study subsets of the phase space with chaotic dynamics. There are several algorithms for finding the dimension d using data from the time series. The calculated dimension of (3.1) for chaotic attractors turned out to be a fractional value, which made it possible to treat such attractors as fractal objects. The concept of strange attractor introduced by D. Ruelle and F. Takens disproved the Landau-Hopf theory of turbulence, where chaotic motion was associated with the trajectory moving along an infinite torus, i.e., the smooth manifold.

Further, as the parameter a increases, each stable point $\psi^3(R)$ experiences a period doubling bifurcation cascade and moves to chaos. However, the domain of chaotic mode that follows the doublings of the cycle of period 3 is represented by three disjoint bands in the phase space. What is different about the behavior of the trajectory is that it gets into these bands strictly periodically. Then, the window of periodicity "closes" for the value of the parameter a_c when three chaotic bands are combined at the instant that coincides with the intersection of the unstable cycle that emerges for the tangent bifurcation and chaotic subsets. When unstable fixed points $\psi^3(R)$ get inside the chaotic bands, the dimensions of the compound chaotic attractor increases dramatically. Aperiodic points of the trajectory immediately emerge in the domains that separate chaotic bands, generating a single attractor that exists in such form till the next tangent bifurcation associated with $\psi^4(R)$.

Thus, dynamics of functional iterations frequently used in simulation can involve complex nonlinear effects that are not taken into account at the stage of designing computer models yet affect simulation results.

4. DESIGNING THE MODEL FOR STUDYING STEPWISE CHANGES

The mathematical part of the theory of biological population recruitment required a qualitative extension through synthesis with ideas of other biological theories that developed in parallel. The conceptual extension of knowledge implies that the applied mathematical apparatus is modified respectively. Dynamic models based on the apparatus of discrete mappings frequently fail to adequately describe characteristic changes of stages of the process since they are a simplified and approximate formalization of the dependence that does not take into account possible conditions for abrupt metamorphoses of different nature.

The need to study systems with critical stepwise changes in their development requires more advanced and detailed type of models to be developed. Such systems are common in simulation of technological processes, consequences of joint functioning of continuous plants and discrete controllers such as an oscillating loop subject to a piecewise-constant action. A similar problem arises for simulation of the flying vehicle moving in the atmosphere and deals with the fact that the air resistance and velocity of sound depend on the altitude, which means we need to provide for altitude ranges, where different dependences hold.

As applied to natural processes and as was shown in [10], the influence of metamorphoses in early ontogenesis of different fish species specifies critical peculiarities of the dependence of the magnitude of mature generation on the recovery conditions. A technique of applying discontinuous differential equations is developed for simulation problems of such processes. For special states attained in the space of state variables, values of parameters in right-hand sides, the form of the right-hand side or the number of equations can change. The algorithmic model uses predicates to describe events. The predicates single the event out of all system states that changes the nature of process development. The way time is represented in the computer simulation environment is the key peculiarity of this technique [11].

Hybrid time is implemented in the simulating algorithm based on numbered and ordered sequence of frames, which makes it possible for the continuous time component to give its place to discrete samples. Software implementation of hybrid model time is expressed by a set of sequentially placed segments with a "time gap" between the end of the current interval and the beginning of the next interval, where the state variables change

$$\varphi = \{\{Gap\_pre_1, [0, T_1], Gap\_post_1\}_1, \ldots \{Gap\_pre_n, [T_{n-1}, T_n], Gap\_post_n\}_n\}.$$

Here, *Gap_pre* is the time gap used to calculate the consistent initial conditions and check the predicate at the left end of the interval of the next long-term behavior, *Gap_post* is a similar gap used to calculate new initial conditions at the right end of the current interval φ_i; T_i is the time of transition actuation or the point, at which the predicate of the event leading to behavior change becomes true.

Our model leverages the AnyLogic hybrid automaton formalism to implement the search of the condition, under which the right-hand side changes. Biological peculiarities of sturgeon (the main object of our study) imply three evolution stages in the model that are reasonable to be outlined by unit, distinct morphological characters [12]. Transitions between these stages in the early ontogenesis of fish happen once aggressive feeding begins and the tactile contact with the ground is lost. The V.V. Vasnetsov theory of stage-by-stage fish development states that the transition between consecutive stages is fast, with the existence conditions for the organism changing dramatically [13]—this inevitably affects mortality factors, which is critical to the problem involved.

We introduce a new model that takes into account critical stepwise changes in development of organisms, with the state of the model characterized by the principal value N at the instant T, and the recruitment variable is $R = N(T)$. The new dependence is the result of numerical solution of the equation with structurally changing right-hand side

$$\begin{cases} \dfrac{dN}{dt} = \begin{cases} -(\alpha w(t)N(t) + \theta(S)\beta)N(t), & 0 < t \le \tau, \\ -(\alpha_1 N(\tau)/w(\tau) + \beta)N(t), t > \tau, & w(t) < w_{k1}, \\ -(\alpha_2 w(t)N(t))N(t - \varsigma), & w_{k1} < w(t) < w_k. \end{cases} \\ \dfrac{dw}{dt} = \dfrac{g}{N^k(t) + \zeta}, \end{cases} \qquad (4.1)$$

where α, α_1, and α_2 are the values of the successively changing coefficient of compensatory mortality, β is the coefficient of decompensatory mortality, the decreasing function $\theta(S)$ introduced in the equation reflects the Allee effect $\lim\limits_{S \to \infty} \theta(S) = 1$, τ is the duration of the earliest development stage subjected to strong negative environmental influence, and ζ is a delay small as compared to the integration interval $t \in [0, T]$. The second equation of (4.1) gives the dynamics of size development of individuals, where g stands for the available food reserve, ζ is the correction factor, and w_k is interpreted as the organism development level that, when achieved, changes the effect of some factors of compensatory and decompensatory mortality included in the model. The initial conditions of the equations are $N(0) = \lambda S$ and $w(0) = w_0$, where λ is the mean fertility of the population and S is the magnitude of its spawning part.

Using the equation with a divergent argument to describe dynamics of the system moving to the third stage of development of the simulated process is due to the common natural phenomenon: the system state at the given instant depends on the processes that took place during a long interval preceding the current instant. For instance, the state of technical frames depends on previous deformations while food available for the current generation depends on the magnitude of previous generations of the population.

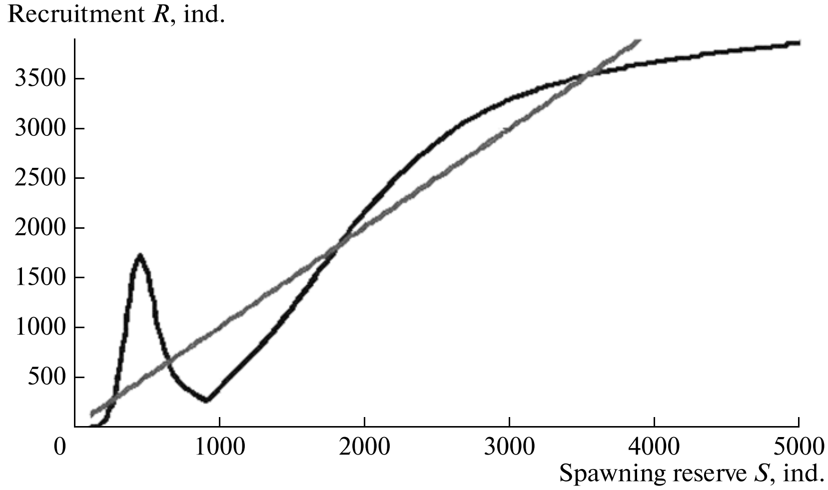

Fig. 3. Functional dependence of the developed model.

5. PROPERTIES OF NONATTRACTING CHAOTIC SET FOR (4.1)

We studied hybrid model (4.1) to see that structural changes makes the dependence between two main variables $N(T) = f(S)$ cease to be described by a unimodal curve. The obtained dependence is characterized by the fact that Schwartz differential invariant (2.1) is not sign-constant since the sign of S_ψ cannot be constant everywhere when there are more than one point of inflexion, at which the curve changes from convex to concave. The dependence gets a complex, wavelike form (Fig. 3) distinguished by the position of the graph with respect to the bisectrix of the coordinate angle. The graph resulted from the numeric study of model (4.1) crosses the bisectrix of the coordinate angle, the geometrical locus of stationary points $R^* = \psi(R^*)$, four times. To analyze stability of fixed points of the dynamic system implemented in the simulation environment, we employed the property of the second iteration $f^2(x)$. Whether the inequalities $f^2(x) > x$ for $x < x^*$ and $f^2(x) < x$ for $x > x^*$ are met is the necessary and sufficient stability condition for the fixed point x^* of the one-dimensional mapping. We can leverage computer simulation to obtain graphs of higher iterations $f(f(...f(x)...))$ of functions that are difficult to be studied analytically.

We analyzed the second iteration $f^2(R)$ to see that, unlike the case of the third iteration $\psi^3(R)$ of the Ricker function when the cycle of period 3 emerges, stable and unstable fixed points do not alternate. First three non-trivial stationary points are unstable, making the trajectories leave their neighborhoods. The dynamic system has two attractors, viz. the fourth singular point R_4^* and the trivial equilibrium (0, 0) with their respective basins of attraction Ω_1 and Ω_2. This means that the simulated system can be in two stable states, transition to one of which means irreversible degradation.

The fact that Ω_1 and Ω_2 do not form any continuous subspaces in the phase space is an even more interesting specific feature of model (4.1). Although the unique singular repeller point $R_r \notin \Omega_1 \cup \Omega_2$ is the boundary between basins of attraction in our model developed previously, this case gave us a simplified idea of the properties of the simulated process, viz. it is a kind of mathematical idealization. Chaos is realized in the basin Ω_3 bounded by the fixed points R_1^* and R_3^*, with aperiodic values of the trajectory points emerging that are never repeated exactly and all approximate repetitions have finite duration. A chaotic subset of the phase space is formed that do not exhibit the properties of attractor. The type of chaotic behavior that takes place in the limited time is called chaotic transient. The number of values, for which the points of the trajectory of the computer model $R_{j+1} = \psi(R_j)$ for $R_0 \in \Omega_3$ are in the chaotic subset Ω_3, i.e., the duration of the chaotic transient mode, significantly depends on the chosen initial conditions R_0.

For smooth boundaries, when the initial conditions were set, the trajectory corresponded to Ω_1 or Ω_2 and a small change in the initial conditions did not make the phase trajectory leave for the alternative attractor. The basins Ω_1 and Ω_2 are sectioned and have a complicated structure. The basin of attraction of the attractor R_4^* is broken in Ω_3 by the segments that belong to the basin of attraction of R_0^*, and the boundaries of basins of attraction are locally disconnected according to the existing classification of complex boundaries of basins of attraction [14].

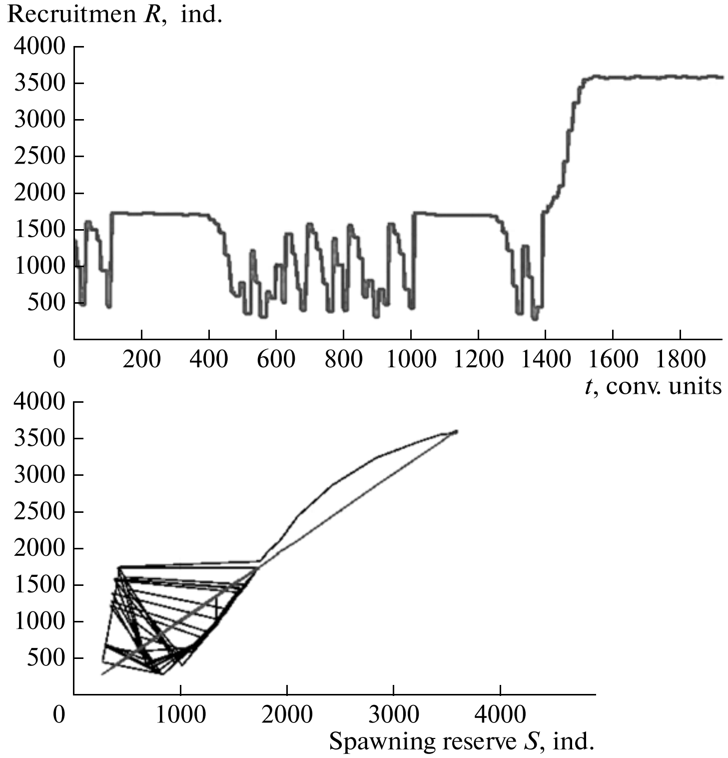

Fig. 4. Chaotic transient (4.1), $\lim_{n \to \infty} \psi^n(R_0) = R_4^*$.

When it leaves Ω_3, the trajectory is sure to go towards one of two possible attractors (Fig. 4). Boundaries of basins of attraction of the type described by the example of the developed model make it impossible to predict what particular attractor of several existing ones will have the phase trajectory in the continuous part of its basin of attraction given some initial state that corresponds to Ω_3. The result may depend on insignificant errors associated with the way numerical methods are implemented and numbers are represented in the computer. Hence, this model exhibits a fundamentally different, as compared to models with chaotic attractor, type of sensitive dependence on initial conditions [15]. We define the described effect arising in computer simulation as uncertainty with respect to asymptotic state. Such uncertainty makes long-term prediction theoretically impossible.

Poincare was the first to mention sensitive dependence on initial conditions that disproved the classical concept of determinism, with the phenomenon studied and evaluated not before researchers could use computers in their work. It was the Royal McBee LGP-30 E. Lorenz used to analyze the model of atmosphere now known as the Lorenz attractor. Although a whole number of other works deal with a way such attractors are formed, in this work we show that a researcher operating with models, where only regular attractors exist, can come across critical nonlinear effects. Such models do not allow a strange attractor to be formed after bifurcation cascade, which is the most common and studied scenario of emergence of deterministic chaos.

CONCLUSIONS

Model (4.1) takes into account dramatic changes in the development of the simulated process that are associated with the rapid increase of the principal parameter observed after the trajectory leaves the chaotic subset for the continuous basin of attraction of the stable state R_4^*. The behavior of the model describes common natural phenomena, in particular sudden outbreaks in the number of some species. Several thousand king crabs were introduced into the Barents Sea at the beginning of 1970s. Their number stayed respectively small for more than thirty years until 1998 when it started increasing rapidly, from 3 to 60 million in 2008 [16].

Outbreak in population numbers is not the only effect. Dramatic and sudden decrease in the number of controlled population is much more feasible. This is what happened with sturgeon fishery. Researches show that these populations are characterized by apparent nonlinear dependence $R = f(S)$. After a period of relative stabilization, Caspian sturgeon take was reduced rapidly and dramatically in 1988–1990 despite the large-scale artificial recovery. Fishery experts did not expect the population size to run through a certain value that corresponded to the repeller point of (4.1), which resulted from exceeding the admissible taking level. This triggered degradation since fishery was not stopped timely. At present, the populations that kept yielding permanent takes during 1970s lost their fishery value, leading to the official fishery ban.

If they ignore all the peculiarities described according to the results of studying the whole class of models, researchers who construct a model for practical problems are destined to make a fortiori wrong conclusions. Underestimating nonlinear effects, they frequently concluded that theoretical basis of the computer model was wrong. It becomes even more complex as the number of bifurcation parameters and the dimension of the phase space grow. It would be too early to disprove theoretical concepts just because the model on the computer display does not behave the way it was expected to. One needs to take a close look at the mathematical apparatus both as functions, viz. finding out if they have extremums and asymptotes, and nonlinear dynamic systems. Studying dynamic systems should go beyond stating that the trajectory is attracted to one attractor, viz. an asymptotically stable stationary state, since nonlinear models can have non-trivial boundaries of basins of attractions.

Analysis of the model may well lead to a conclusion that dynamics of the simulated process is unpredictable, which is far from implying that the model is inadequate or the applied methods are not correct.

REFERENCES

1. V. Volterra, *The Theory of Functionals, Integral, and Integro-Differential Equations* (Nauka, Moscow, 1982) [in Russian].
2. G. Paulic, *Digital Simulation of Nature Communities* (New York, 1964).
3. W. Ricker, "Stock and Recruitment," J. Fisheries Research Board of Canada **11** (5), 559–623 (1954).
4. J. Milnor, "On the Concept of Attractor," Comm. Math. Phys. **99**, 177–195 (1985).
5. E. Ott, *Chaos in Dynamical Systems* (University Press, Cambridge, 1993).
6. A. N. Sharkovskii, "Coexistence of Cycles of a Continuous Transformation of a Straight Line into Itself," Ukr. Mat. Zhurn **26** (1), 61–71 (1964).
7. M. J. Feigenbaum, "Universal Behavior in Nonlinear Systems," Phys. D (Amsterdam) **7** (1), 16–39 (1983).
8. C. Grebogi, E. Ott, and J. A. Yorke, "Fractal Basin Boundaries, Long-Lived Chaotic Transients and Unstable-Unstable Pair Bifurcation," Phys. Rev. Lett. **50** (13), 935–938 (1983).
9. J. Farmer, E. Ott, and J. Yorke, "The Dimension of Chaotic Attractors," Phys. D (Amsterdam) **7**, 153–170 (1983).
10. A. Yu. Perevaryukha, "Simulation of the Effect of a Wave-like Curve of Reproduction of a Fish Population," Ekologicheskie Sistemy i Pribory, No. 8, 41–44 (2008).
11. Yu. B. Kolesov and Yu. B. Senichenkov, *System Simulation. Dynamic and Hybrid Systems* (BKhV, St. Petersburg, 2006) [in Russian].
12. L. A. Sytina and O. B. Timofeev, "Periodization of Development of Sturgeon (Asirenseridae) and the Problem of Variability of Organisms," Voprosy Ikhtiologii **1** (2), 275–289 (1973).
13. O. I. Shmal'gauzen, "Durability and Typicality of Development of Pre-Larva of Beluga and Stellate Sturgeon under Different Temperature Conditions," in *Biologicheskie osnovy osetrovodstva* (Nauka, Moscow, 1983), pp. 95–103 [in Russian].
14. S. MacDonald, C. Grebogi, E. Ott, et al., "Fractal Basin Boundaries," Physica D **17** (2), 125–153 (1985).
15. E. N. Lorenz, "Deterministic Nonperiodic Flow," J. Atmospheric Sciences **20** (2), 130–141 (1963).
16. S. V. Bakanev, "Problems of Evaluation of the Stock and Regulation of Fisheries of Kamchatka Crab in the Barents Sea," Voprosy Rybolovstva, No. 1, 51–63 (2009).